

On longitudinal motion in a magnetic field

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An exact solution is presented of the equations and boundary conditions governing the steady longitudinal motion of a semi-infinite non-conducting plate in an oblique magnetic field. The discussion covers the distortion of the boundary layer, the structure of the induced electric and magnetic fields, the current-density distribution, and the behaviour of the fields near the edge of the plate.

1. Introduction

The steady longitudinal motion of an infinitely long cylinder in an unbounded viscous conducting fluid has recently been considered by Hasimoto (1960). (Longitudinal motion refers to motion parallel to the generators of the cylinder.) If the applied magnetic field is perpendicular to the axis of the cylinder, the fundamental magnetohydrodynamic equations are linear and more amenable to analysis. In particular, an interesting explicit solution was presented in the aforementioned paper, for the special case of a semi-infinite plate moving in a conducting fluid with a magnetic field directed perpendicular to the plate.

In this paper, we consider a generalization of this problem in which a rigid *non-conducting* plate is moved with constant velocity in an obliquely incident uniform applied magnetic field. (Viscosity and electrical conductivity are assumed uniform and constant.) Although the motion which develops is no longer symmetrical with respect to the plane of the plate (the x -axis), it is still possible to solve the problem explicitly in terms of elementary known functions. Many interesting features such as the behaviour of the magnetic field near the edge of the plate and the structure of the boundary layer can then be studied in exact detail.

2. Theory

Consider the impulsive longitudinal motion of a semi-infinite flat plate in the configuration shown in figure 1. Initially, the induced electromotive force, $\mathbf{q} \times \mathbf{H}_0$, drives a current 'into' the plate, i.e. the component of current normal to the plate is non-zero. Since the plate is a non-conductor, charge separation occurs. The plate acquires a double-layer dipole-charge distribution and the fluid acquires a volume-charge density. As a consequence, the electric field which is thus created turns the current away from the boundary thereby reducing the normal component of the current at the plate to zero. Off the plate, however, electric currents can, and do, flow from one half-plane to the other.

The disturbances produced in the flow at and near the plate are propagated along the magnetic field lines by Alfvén waves, diffusing, in the process, because

of viscosity and electrical conductivity. Since the Alfvén waves propagating along those field lines which intersect the plate, arise in the regions of greatest vorticity and current generation, the ‘steady’ configuration should show that the disturbance is concentrated predominantly in a skewed right half-plane. We expect then, that the line $x = y \cot \alpha$ divides the plane into a region of intense disturbance and a region of relative quiescence.

Furthermore, we expect that the disturbances appearing between the applied field lines L_1 and L_2 (a distance d apart) are in some way proportional to the length of plate, $d \operatorname{cosec} \alpha$, which these lines intercept.

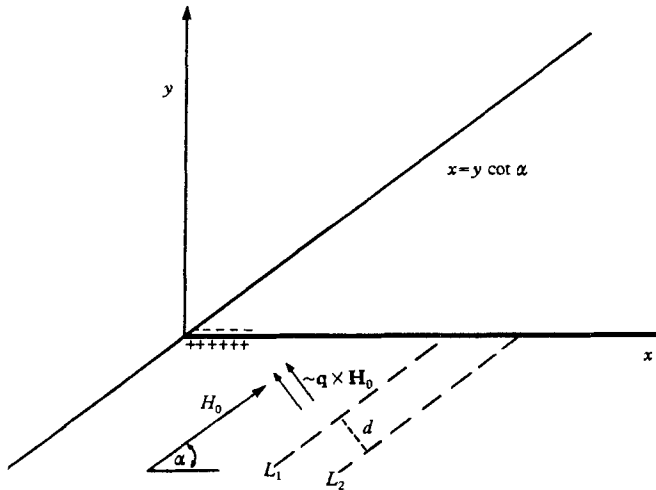


FIGURE 1. Basic configuration.

We can then infer that there is probably no steady solution in which the induced quantities are zero at infinity ($-x \rightarrow \infty$), if the magnetic field is directed parallel to the plate, $\alpha = 0$. In this case, the projected plate length is infinite, indicating that the disturbances contained between these field lines arise in the entire boundary layer and may then be of sufficient strength to produce non-zero perturbations everywhere at infinity. Although the argument is crude, we will verify this conclusion in a subsequent paragraph.

The steady flow we refer to is that which would occur if no Alfvén waves reflect from the outer boundaries and return to the finite domain. If R is the distance to the outer boundary and t is the time, the conceptual limit process is $\lim_{t \rightarrow \infty} \lim_{R \rightarrow \infty}$. This limiting procedure is of course non-uniform and it has been shown,

Carrier & Greenspan (1959), that the reverse procedure ($\lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty}$) may result in a

different steady-state solution. The effects of the reflected Alfvén waves can indeed be important. In any event, the ‘steady’ solution derived here will be valid in any finite domain after the passage of the initial wave train and before the return of their reflexions from the outer boundaries.

The induced currents, in the problem under consideration, lie in the plane of the applied field and thereby constitute infinitely long solenoids of current. The induced magnetic field is then parallel to the edge of the plate and in particular

is zero on the plate. The fluid velocity is also in the direction of motion and all quantities are functions of the space co-ordinates x and y alone. As a consequence $\nabla \cdot \mathbf{q} = 0$ and the flow is incompressible. For simplicity, however, we will assume that the density is constant although the results are valid for a more general situation.

The basic equations governing the flow are

$$\begin{aligned} (\mathbf{q} \cdot \nabla) \mathbf{q} &= -\frac{1}{\rho} \nabla P + \mu(\mathbf{j} \times \mathbf{H}) + \nu \nabla^2 \mathbf{q}, \\ \nabla \cdot \mathbf{q} &= 0, \\ \nabla \times \mathbf{H} &= \mathbf{j} = \sigma(\mathbf{E} + \mu \mathbf{q} \times \mathbf{H}), \\ \nabla \cdot \mathbf{H} &= \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{E} &= \rho_e. \end{aligned}$$

Since $\mathbf{H} = H_0 \cos \alpha \hat{\mathbf{i}}_1 + H_0 \sin \alpha \hat{\mathbf{i}}_2 + h(x, y) \hat{\mathbf{i}}_3,$
 $\mathbf{q} = v(x, y) \hat{\mathbf{i}}_3, \quad \mathbf{E} \cdot \hat{\mathbf{i}}_3 = \mathbf{j} \cdot \hat{\mathbf{i}}_3 = 0,$

these equations can be reduced to

$$\nu \nabla^2 v + \frac{\mu}{\rho} H_0 (h_x \cos \alpha + h_y \sin \alpha) = 0$$

and $\frac{1}{\mu \sigma} \nabla^2 h + H_0 (v_x \cos \alpha + v_y \sin \alpha) = 0.$

In terms of dimensionless (primed) quantities representing length, $l = \nu(\epsilon\beta)^{-\frac{1}{2}} V_0^{-1} l'$; velocity, $v = V_0 v'$; magnetic field $h = H_0 \epsilon^{\frac{1}{2}} \beta^{-\frac{1}{2}} h'$; and electric field $E = \mu v_0 H_0 E'$, where $\epsilon = \mu\nu\sigma$ and $\beta = \mu H_0^2 / \rho V_0^2$, these equations become (upon dropping the primes)

$$\nabla^2 v + (h_x \cos \alpha + h_y \sin \alpha) = 0, \tag{2.1}$$

$$\nabla^2 h + (v_x \cos \alpha + v_y \sin \alpha) = 0. \tag{2.2}$$

The boundary conditions are

$$\left. \begin{aligned} h(x, 0) &= 0, \\ v(x, 0) &= 1, \end{aligned} \right\} (x \geq 0)$$

with $\lim_{x \rightarrow -\infty} v(x, y) = \lim_{x \rightarrow -\infty} h(x, y) = 0.$

The first condition, $h = 0$ for $x \geq 0$, requires further explanation.

Consider a plate of finite length $2l$, and infinitesimal thickness, 2δ , with the origin of the x, y co-ordinate system located at its centre as shown in figure 2. Since the plate is an insulator $\mathbf{j} \cdot \hat{\mathbf{n}} = 0 = \partial h / \partial s$, on its surface, the contour C . (In the limit, as δ becomes zero, this condition is $\{\partial h(x, 0)\} / \partial x = 0$ for $|x| \leq l$.) Therefore, the induced field has a constant value on (and inside) the plate. In order to determine this constant, another condition must be utilized and it is easily shown (Hasimoto, 1960) that a further requirement is

$$\oint_{C_+} \frac{\partial h}{\partial n} ds = 0,$$

where C_+ is the external perimeter. (This constraint may be deduced in the following way. Since the electric field is irrotational,

$$\oint_{C_-} \mathbf{E} \cdot d\mathbf{s} = 0.$$

However, the tangential component of the electric field is continuous across the plate so that

$$\oint_{C_+} \mathbf{E} \cdot d\mathbf{s} = 0$$

or in terms of the magnetic field

$$\frac{1}{\sigma} \oint_{C_+} \nabla \times \mathbf{H} \cdot d\mathbf{s} - \oint_{C_+} \mu \mathbf{q} \times \mathbf{H} \cdot d\mathbf{s} = 0.$$

The second integral is zero because the integrand is a constant on the contour C_+ . The first then reduces to the foregoing condition.)

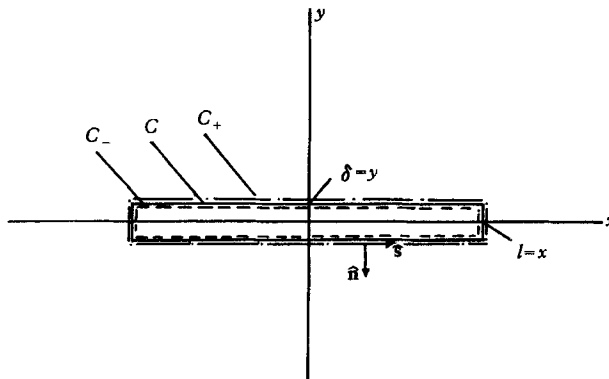


FIGURE 2. The finite flat plate.

In particular, as $\delta \rightarrow 0$, this condition becomes

$$\int_{-l}^l \left(\left. \frac{\partial h}{\partial y} \right|_{0+} - \left. \frac{\partial h}{\partial y} \right|_{0-} \right) dx = 0.$$

(The induced charge distributions on the upper and lower surfaces coalesce to form a dipole layer.)

We must now prove that the solution (for $\delta = 0$) which satisfies the boundary condition $h(x, 0) = 0$ for $|x| \leq l$ also satisfies this additional constraint.

Let $v^{(0)}(x, y)$ and $h^{(0)}(x, y)$ be solutions of the basic equations (2.1) and (2.2) which vanish at infinity and assume the values $v^{(0)}(x, 0) = 1$, $h^{(0)}(x, 0) = h_0$ (an arbitrary constant) on the plate, $|x| \leq l$. It is easily verified that the functions

$$\begin{aligned} w(x, y) &= v^{(0)}(-x, -y), \\ g(x, y) &= -h^{(0)}(-x, -y) \end{aligned}$$

are also solutions of the fundamental equations which assume the values $w = 1$ and $g = -h_0$ on the plate. Using the principle of superposition, we obtain the solutions

$$\begin{aligned} v(x, y) &= \frac{1}{2}[v^{(0)}(x, y) + v^{(0)}(-x, -y)], \\ h(x, y) &= \frac{1}{2}[h^{(0)}(x, y) - h^{(0)}(-x, -y)], \end{aligned}$$

which have the properties

$$\begin{aligned} v(x, y) &= v(-x, -y), \\ h(x, y) &= -h(-x, -y), \\ v(x, 0) &= 1, \quad h(x, 0) = 0 \quad \text{for } |x| \leq l. \end{aligned}$$

The function h , so defined satisfies the integral constraint and is zero on the plate. To prove this, we rewrite the integral condition as

$$\int_0^l [h_2(x, 0+) - h_2(x, 0-)] dx + \int_0^l [h_2(-x, 0+) - h_2(-x, 0-)] dx = 0,$$

where the numerical subscript two denotes differentiation with respect to the second variable, i.e. $h_2(x, y) = h_y(x, y)$. From the definition of $h(x, y)$, however, it follows that

$$\begin{aligned} h_2(x, 0+) &= h_2(-x, 0-); \\ h_2(x, 0-) &= h_2(-x, 0+). \end{aligned}$$

Substituting these values in the preceding equation reduces it to an identity proving the proposition. Therefore, the unique solution for which $h = 0$ on the plate also satisfies the addition integral condition. Since the semi-infinite plate is but an idealization of a long finite plate it is evident that $h(x, 0) = 0$ for $x \geq 0$ is the correct boundary condition.

Physically, this fact that the induced field is zero on the plate implies that there are no closed current paths containing the plate. It is known that this is the case when $\alpha = \frac{1}{2}\pi$. The effect of the oblique field is to stretch and distort current lines as shown in figure (5).

We can now proceed to solve the boundary-value problem, equations (2.1), (2.2) et seq. Let

$$\begin{aligned} \phi &= v + h, \\ \psi &= v - h, \end{aligned}$$

then
$$\nabla^2 \phi + \phi_x \cos \alpha + \phi_y \sin \alpha = 0, \tag{2.3}$$

with
$$\phi(x, 0) = 1 \quad \text{on } x \geq 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \phi = 0;$$

and
$$\nabla^2 \psi - (\psi_x \cos \alpha + \psi_y \sin \alpha) = 0, \tag{2.4}$$

with
$$\psi(x, 0) = 1 \quad \text{on } x \geq 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \psi = 0.$$

We now have two completely independent problems to solve.

In the particular case $\alpha = 0$, equation (2.3) is identical with that governing conduction of heat in a fluid flowing with unit negative velocity from $x = +\infty$ past a semi-infinite plate with unit temperature extending between $0 \leq x < \infty$. It is evident that under these conditions the temperature in the entire fluid region is unity so that $\lim_{x \rightarrow -\infty} \phi \neq 0$. The case $\alpha = 0$ is then distinguished from all others, in that a solution of the foregoing boundary-value problem does not exist.*

* The problem is still equivalent, after a simple variable substitution, to that for heat transfer with a negative velocity field; however, the plate temperature is $\exp[-x(1 - \cos \alpha)/2]$ and the solution exists.

The solution of equations (2.1) and (2.2) is

$$v = \frac{1}{2}(\phi + \psi), \quad h = \frac{1}{2}(\phi - \psi), \tag{2.5}$$

where

$$\phi = \Phi(x, y) \exp(-\frac{1}{2}y \sin \alpha), \quad \psi = \Psi(x, y) \exp(\frac{1}{2}y \sin \alpha), \tag{2.6}$$

$$\begin{aligned} \Psi = \frac{1}{2}[\exp(\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\eta \cos \frac{1}{2}\alpha + \xi \sin \frac{1}{2}\alpha) \\ + \exp(-\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\eta \cos \frac{1}{2}\alpha - \xi \sin \frac{1}{2}\alpha)], \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Phi = \frac{1}{2}[\exp(\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\eta \sin \frac{1}{2}\alpha + \xi \cos \frac{1}{2}\alpha) \\ + \exp(-\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\eta \sin \frac{1}{2}\alpha - \xi \cos \frac{1}{2}\alpha)] \end{aligned} \tag{2.8}$$

and $(\xi + i\eta)^2 = x + i|y| = r \exp i\theta. \tag{2.9}$

A method of deriving these formulas is presented in the Appendix. It is now easily verified that the solution does indeed degenerate for $\alpha = 0$.

From these formulas, it can be shown that the viscous skin friction is

$$v_y(x, 0+) - v_y(x, 0-) = [v_y]_0^{\pm} = -\frac{1}{2} \sin \alpha [G(x^{\frac{1}{2}} \sin \frac{1}{2}\alpha) + G(x^{\frac{1}{2}} \cos \frac{1}{2}\alpha)] \tag{2.10}$$

and that the discontinuity in the tangential component of current density is

$$[h_y]_0^{\pm} = \frac{1}{2} \sin \alpha [G(x^{\frac{1}{2}} \sin \frac{1}{2}\alpha) - G(x^{\frac{1}{2}} \cos \frac{1}{2}\alpha)]. \tag{2.11}$$

Here $G(x) = \operatorname{erf} x + \frac{1}{\pi^{\frac{1}{2}}} \exp(-x^2). \tag{2.12}$

Equation (2.11) shows that the tangential component of the electric field is discontinuous across the plate. There must then be a double layer (dipole) distribution, of strength τ , on this surface (see Stratton (1941), page 191) given by

$$-\nabla \tau = \mathbf{E}_+ - \mathbf{E}_-,$$

or $\frac{d\tau}{dx} = [h_y]_0^{\pm} \tag{2.13}$

The last equation can be integrated and the result is

$$\tau(x) = \frac{1}{2} \sin \alpha \left[\frac{F(x^{\frac{1}{2}} \sin \frac{1}{2}\alpha)}{\sin^2 \frac{1}{2}\alpha} - \frac{F(x^{\frac{1}{2}} \cos \frac{1}{2}\alpha)}{\cos^2 \frac{1}{2}\alpha} \right] \tag{2.14}$$

with $F(x) = x^2 \operatorname{erf} x + \pi^{-\frac{1}{2}} x \exp(-x^2) + \frac{1}{2} \operatorname{erf} x.$

At large distances from the edge of the plate, the following formulas hold ($x + i|y| = r \exp i\theta$);

$$\left. \begin{aligned} v &\sim \frac{1}{2}(1 + \exp -|y| \sin \alpha) \\ h &\sim -\frac{1}{2} \frac{y}{|y|} (1 - \exp -|y| \sin \alpha) \end{aligned} \right\} \quad (0 \leq \theta < \alpha),$$

$$v \sim h \sim \frac{1}{2} \exp[-\frac{1}{2}(y + |y|) \sin \alpha] \quad (\alpha < \theta < \pi - \alpha),$$

$$v \simeq h \simeq 0 \quad (\pi - \alpha < \theta < \pi).$$

It is seen that the line, $x = y \cot \alpha$, does indeed separate the plane into a region of intense disturbance, $x > y \cot \alpha$, and a region of relative quiescence $x < y \cot \alpha$. Along this dividing line $v \sim h \sim \frac{1}{4}$ on $\theta = \pi - \alpha$, $y < 0$ and $v \sim -h \sim \frac{1}{4}$ along $\theta = \alpha$, $y > 0$. The situation is summarized in figure 3.

Figure 4 shows the lines of constant value of the induced magnetic field, which are also the current paths, for the angle $\alpha = \frac{1}{2}\pi$. The behaviour of the field or current near the edge of the plate is interesting. The zero field line ($h = 0$), which

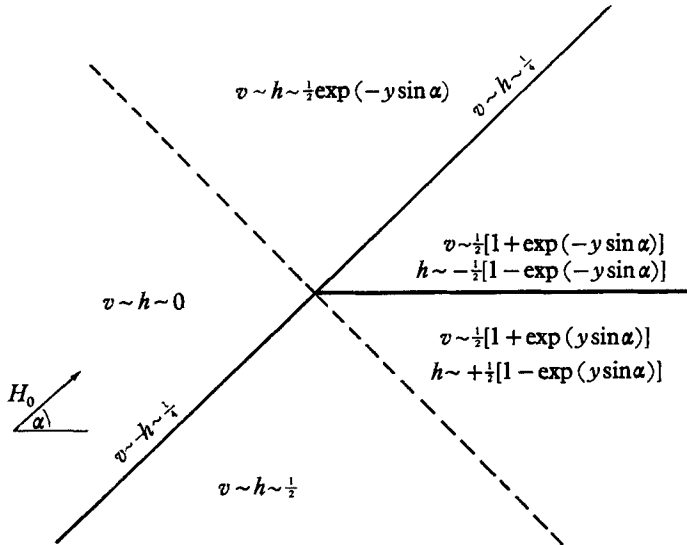


FIGURE 3. Asymptotic values of the magnetic field and fluid velocity at large distances from the edge of the plate.

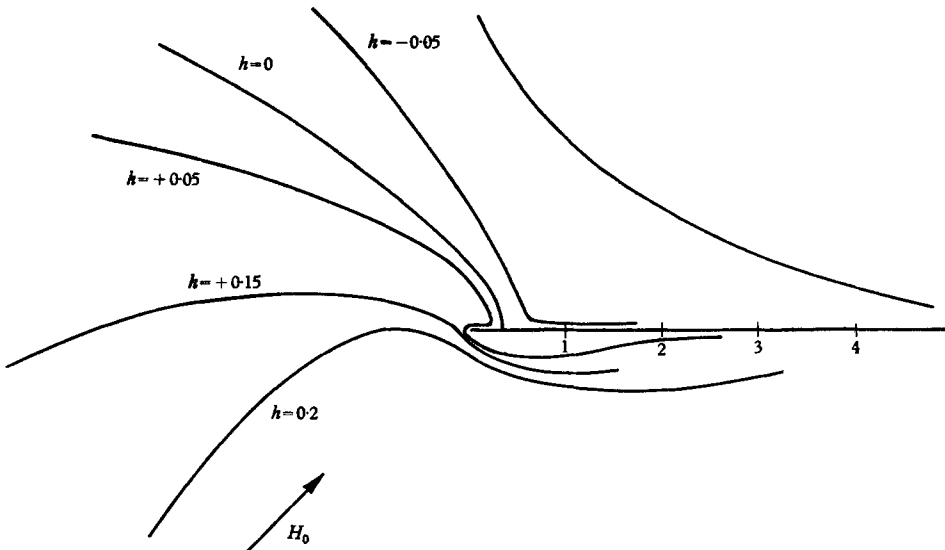


FIGURE 4. The magnetic field (or current lines) induced by the longitudinal motion of an insulating plate in an oblique applied magnetic field.

is the real axis for $\alpha = \frac{1}{2}\pi$, is here distorted into the upper half-plane. One branch of the line is, of course, still the plate; the other emanates from the plate a finite distance from the edge and then continues into the second quadrant.

The point on the plate, x_0 , at which this separation occurs is determined from the condition $j = 0$, and it is found that

$$G(x_0^{\frac{1}{2}} \sin \frac{1}{2}\alpha) - G(x_0^{\frac{1}{2}} \cos \frac{1}{2}\alpha) = 2.$$

The separation distance increases to infinity as α approaches zero. The quantities, $v_{yy}(x, 0+)$, $h_y(x, 0+)$, change sign from plus to minus as x becomes larger than x_0 . In the vicinity of the edge, $r \ll 1$;

$$\begin{aligned} h &\simeq \pi^{-\frac{1}{2}}\eta (\cos \frac{1}{2}\alpha - \sin \frac{1}{2}\alpha), \\ v &\simeq 1 - \pi^{-\frac{1}{2}}\eta (\cos \frac{1}{2}\alpha + \sin \frac{1}{2}\alpha). \end{aligned}$$

In general, current from the lower half plane penetrates into the upper and returns to close the loop. The motion of a long finite plate will also show this behaviour at the edges and the probable current distribution is shown in figure 5.

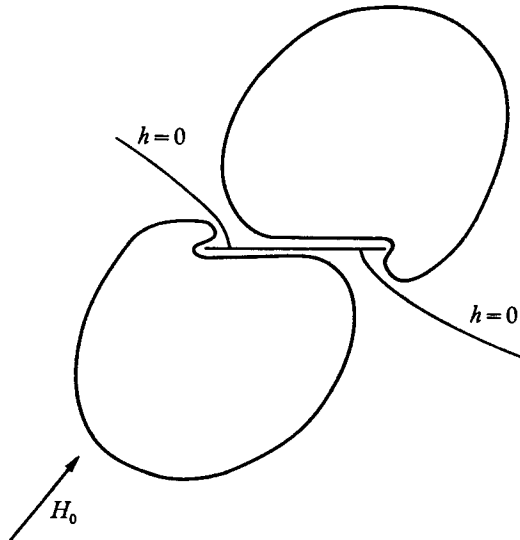


FIGURE 5. The induced magnetic field or current lines for a long plate of finite length.

Figure 6 illustrates the distortion of the boundary layer. The disturbance behind the dividing line is accentuated by the Alfvén waves propagating along the field lines intersecting the plate.

3. Conclusions

An exact solution has been presented herein, which describes the longitudinal motion of a semi-infinite non-conducting plate in an oblique magnetic field. It is found that the plate becomes electrically charged by acquiring a double-layer distribution. The induced electric field turns the current away from the plate thereby reducing the normal component of current to zero at the non-conducting surface.

The disturbances are propagated into the fluid by Alfvén waves. These waves move along the magnetic field lines; consequently, the predominant effects are found in the half-plane in which the magnetic field lines intersect the plate

(the area of greatest disturbance). The fluid boundary layer is distorted and stretched along the field lines as is the induced magnetic field. Electric currents arising in the lower half-plane penetrate the upper half-plane. One branch of the curve of zero field strength is of course the plate; the other branch emanates from the plate a finite distance from the edge and continues into the second quadrant.

The behaviour of the solution near the edge should be of use in any problem involving the motion of a body with sharp corners in an oblique field.

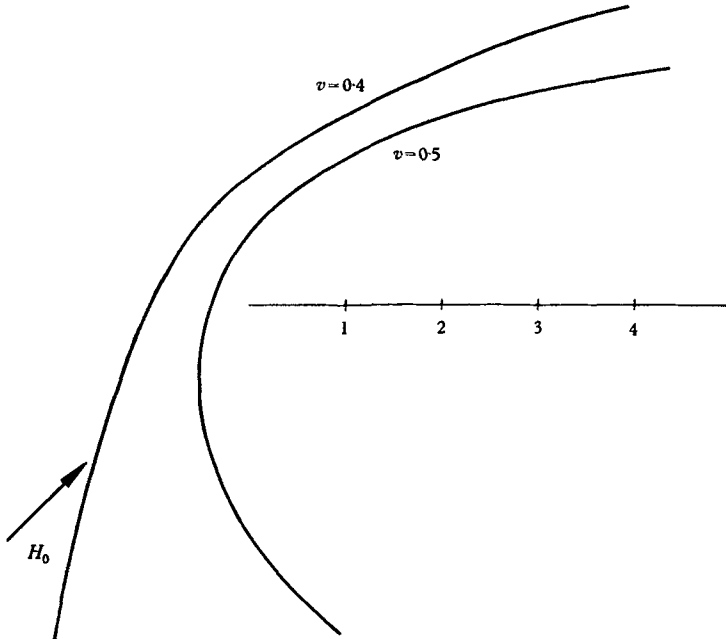


FIGURE 6. The velocity field produced by the longitudinal motion of a plate in an oblique applied magnetic field.

Appendix

The easiest way to solve equations (2.3) or (2.4) subject to the given boundary conditions is, of course, to guess the answer and then to verify it. A more direct approach, requiring less ingenuity, is based on the Wiener-Hopf method. We will briefly sketch the manner in which equation (2.3) is solved; no new techniques are needed to determine the entire solution given in equations (2.7) and (2.8).

Let $\phi = \Phi \exp(-\frac{1}{2}y \sin \alpha)$, then equation (2.3) becomes

$$\nabla^2 \Phi + \Phi_x \cos \alpha - \frac{1}{4} \Phi \sin^2 \alpha = 0$$

with $\Phi(x, 0) = 1$ and $\Phi \rightarrow 0$ as $x \rightarrow -\infty$.

If we define the function

$$f(x) = \Phi_y(x, 0+) - \Phi_y(x, 0-)$$

and the Fourier transform of an arbitrary function $G(x)$ by

$$\bar{G}(p) = \int_{-\infty}^{\infty} G(x) \exp ipx dx,$$

then it is found that

$$\Phi(p, y) = -\frac{1}{2m} \bar{f}(p) \exp -m|y|,$$

where

$$m = [(p + \frac{1}{2}i \cos \alpha)^2 + \frac{1}{4}]^{\frac{1}{2}}.$$

Since $f(x) = 0$ for $x \leq 0$ and $\Phi(x, 0) = 1$, $x \geq 0$, the usual arguments of the Wiener-Hopf method can be employed to determine $\bar{f}(p)$ from the last equation with $y = 0$. The result is

$$\bar{f}(p) = -2i(-\frac{1}{2}i(1 - \cos \alpha))^{\frac{1}{2}}(p + \frac{1}{2}i(1 + \cos \alpha))^{\frac{1}{2}}(p + i\delta)^{-1}$$

with $\delta \rightarrow 0+$. (An analysis similar to this is contained in Greenspan & Carrier (1959).) Therefore

$$\Phi(x, y) = \frac{i}{2\pi} (-\frac{1}{2}i(1 - \cos \alpha))^{\frac{1}{2}} I(x, y),$$

where
$$I(x, y) = \int_{-\infty}^{\infty} \frac{[\exp(-ipx - m|y|)](p - \frac{1}{2}i(1 - \cos \alpha))^{\frac{1}{2}}}{p + i\delta} dp.$$

If we perform the following sequential operations we can evaluate this integral:

- (i) change the variable of integration to $u = p + \frac{1}{2}i \cos \alpha$;
- (ii) assume $x > 0$, and deform the contour into the path $Imu = 0$;
- (iii) change the variable to $u = \frac{1}{2} \sinh \lambda$;
- (iv) deform the contour in the λ plane into the path $Im\lambda = -\frac{1}{2}\pi$;
- (v) change the variable to $\zeta = \lambda + \frac{1}{2}\pi i$;
- (vi) change the variable to $t = \exp \zeta$;
- (vii) use the known result

$$\int_0^{\infty} \frac{1}{t+a} \exp\left(-\frac{\sigma}{t} - \rho t\right) \frac{dt}{t^{\frac{1}{2}}} = \pi a^{-\frac{1}{2}} \exp\left(\frac{\sigma}{a} + a\rho\right) \operatorname{erfc}\left((a\rho)^{\frac{1}{2}} + \left(\frac{\sigma}{a}\right)^{\frac{1}{2}}\right).$$

The result is

$$I = -2\pi i(-\frac{1}{2}i(1 - \cos \alpha))^{-\frac{1}{2}}[\exp(-\frac{1}{2}|y| \sin \alpha) + J],$$

where

$$J = \frac{1}{2}[\exp(\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\xi \cos \frac{1}{2}\alpha + \eta \sin \frac{1}{2}\alpha) + \exp(-\frac{1}{2}|y| \sin \alpha) \operatorname{erfc}(\xi \cos \frac{1}{2}\alpha - \eta \sin \frac{1}{2}\alpha)]$$

and $(\xi + i\eta)^2 = x + i|y|$. The remainder of the solution is completed in essentially the same way.

It should be noted that several other problems involving unsteady motion on perfectly conducting plates are also soluble in this manner.

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